POYNTING THEOREM :- / Imp

The rate of Energy transport per unit area, is called Poynting vector. It is also termed as instaneous energy flux density and is represented by S or P.

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

$\vec{S}$ is perpendicular to both $\vec{E}$ and $\vec{H}$.

Unit - W/m^2

Derivation:- We can calculate the energy density carried by Electromagnetic waves with the help of Maxwell's Equation given below-

i) $\nabla \cdot \vec{B} = 0$ — ①

ii) $\nabla \cdot \vec{B} = 0$ — ②

iii) $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$ — ③

iv) $\nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$ — ④

Now Taking (.) Dot product of ~~③ with~~ H with ③ and Dot product of E with ④

$$\vec{H} \cdot (\nabla \times \vec{E}) = -\mu \vec{H} \cdot \frac{d\vec{H}}{dt} \quad —⑤ \quad (B = \mu \vec{H})$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \frac{d\vec{E}}{dt} \quad —⑥ \quad (D = \epsilon \vec{E})$$

Substracting Equation ⑤ from Equation ⑥

$$\Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{H} (\vec{\nabla} \times \vec{E}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \frac{d\vec{E}}{dt} - \left(-\mu \vec{H} \frac{d\vec{H}}{dt}\right)$$

Since $\vec{A} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \cdot (\vec{B} \times \vec{A}) = -\vec{\nabla} (\vec{A} \times \vec{B})$

$\Rightarrow \cancel{-\nabla\cdot(E\times H) = J\cdot E + \frac{1}{2}\epsilon\frac{d^2E}{dt^2}}$

$$\Rightarrow -\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{J}\cdot\vec{E} + \frac{1}{2}\epsilon\frac{dE^2}{dt} + \frac{1}{2}\mu\frac{dH^2}{dt}$$

using $\left(\because \begin{array}{l} \epsilon\vec{E}\frac{dE}{dt} = \frac{1}{2}\epsilon\frac{dE^2}{dt} \\ \mu\vec{H}\frac{dH}{dt} = \frac{1}{2}\mu\frac{dH^2}{dt} \end{array} \right\}$

$$\Rightarrow -\vec{\nabla}\cdot\vec{S} = \vec{J}\cdot\vec{E} + \frac{1}{2}\frac{d}{dt}(\epsilon E^2 + \mu H^2)$$

or $$\vec{J}\cdot\vec{E} + \frac{d}{dt}\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) + \vec{\nabla}\cdot\vec{S} = 0$$

Taking volume Integral over the volume V enclosed by the surface S,

$$\int_V \vec{\nabla}\cdot\vec{S}\,dv + \frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv = -\int_V \vec{J}\cdot\vec{E}\,dv$$

using Divergence Theorm

$$\boxed{\int_S \vec{S}\cdot d\vec{S} + \frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv = -\int_V \vec{J}\cdot\vec{E}\,dv}$$

→ $\int_S \vec{S}\cdot d\vec{S}$ → Represent rate of flow of Energy or Power flux.

→ $\frac{d}{dt}\int_V\left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right)dv$ = Rate of change of total Energy

→ $\frac{\epsilon E^2}{2}$ & $\frac{\mu H^2}{2}$ represent Energy stored in Electric & Magnetic Field.

$\int_V \vec{J}\cdot\vec{E}\,dv$ = Rate of work done by the Electromagnetic field in displacing the charge within the volume V.

Hence

$$\vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E} + \frac{d}{dt}\left(\frac{1}{2}\mu H^2\right) + \frac{d}{dt}\left(\frac{1}{2}\epsilon E^2\right) = 0$$

OR

$$\int_V \vec{\nabla}\cdot\vec{S}\,dv + \int_V \vec{J}\cdot\vec{E}\,dv + \frac{d}{dt}\int_V \left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2}\right)dv = 0$$

Above Equation is known as Poynting Theorm or work-Energy Theorm.

According to Poynting theorm the power transformed into Electromagnetic field is Equal to the sum of the time rate of change of EM energy within certain volume and the time rate of Energy flowing out through the boundary surface. This is also known as Energy conservation law in Electromagnetism.