Chapter One

Functions and their Graphs

Definition: If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that ***y is a function of x***.

Or, A **function** f is a rule that assigns to each element in a set D exactly one element, called $f(x)$, in a set .

Definition: A ***function*** f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$ (read "f of x").

Independent and Dependent Variables: For a given input x, the output of a function f is called the ***value*** of f at x or the ***image*** of x under f . Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x; the variable x is called the ***independent variable*** (or ***argument***) of f , and the variable y is called the ***dependent variable*** of f . For now we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a ***real-valued function of a real variable***.

Domain and Range: If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the ***domain*** of f , and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f .

Example-1 Find the natural domain of the following functions:

(a) $f(x) = x^3$ (b) $f(x) = 1/[(x-1)(x-3)]$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Solution (d). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

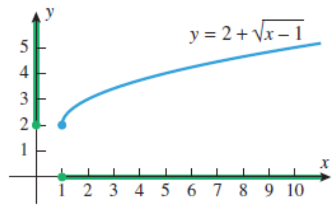
$$x^2 - 5x + 6 = (x-3)(x-2) \geq 0$$

This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty) \blacktriangleleft$$

Example-2 Find the domain and range of

$$\text{(a)}\ f(x) = 2 + \sqrt{x-1} \qquad \text{(b)}\ f(x) = (x+1)/(x-1)$$

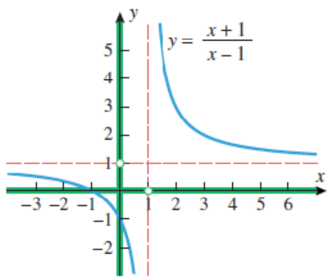


▲ Figure 0.1.15

Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain, $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are highlighted in green on the x- and y-axes in Figure 0.1.15.

Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$



▲ Figure 0.1.16

To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x+1}{x-1} \tag{4}$$

Although the set of possible y-values is not immediately evident from this equation, the graph of (4), which is shown in Figure 0.1.16, suggests that the range of f consists of all y, except $y = 1$. To see that this is so, we solve (4) for x in terms of y:

$$\begin{aligned}(x-1)y &= x+1 \\ xy - y &= x+1 \\ xy - x &= y+1 \\ x(y-1) &= y+1 \\ x &= \frac{y+1}{y-1}\end{aligned}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically. ◀

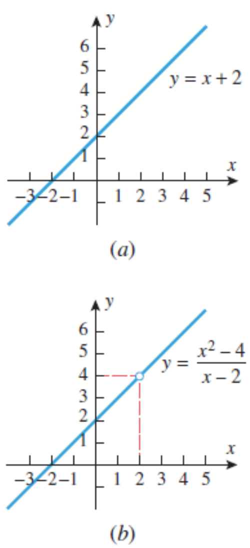
Example-3 The natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

consists of all real x except $x = 2$. However, if we factor the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2 \tag{3}$$

Since the right side of (3) has a value of $f(2) = 4$ and $f(2)$ was undefined in (2), the algebraic simplification has changed the function. Geometrically, the graph of (3) is the line in Figure 0.1.14*a*, whereas the graph of (2) is the same line but with a hole at $x = 2$, since the function is undefined there (Figure 0.1.14*b*). In short, the geometric effect of the algebraic cancellation is to eliminate the hole in the original graph. ◀



▲ Figure 0.1.14

Sometimes alterations to the domain of a function that result from algebraic simplification are irrelevant to the problem at hand and can be ignored. However, if the domain must be preserved, then one must impose the restrictions on the simplified function explicitly. For example, if we wanted to preserve the domain of the function in Example 7, then we would have to express the simplified form of the function as

$$f(x) = x + 2, \quad x \neq 2$$

Example-4 Find the natural domain and range of the functions:

(i) $f(x)=\sqrt{x+2}$ (ii) $f(x)=\sqrt{2-x}$ (iii) $f(x)=\sqrt{x-1}+\sqrt{5-x}$ (iv) $f(x)=|x+3|$

(v) $f(x)=2+|x-2|$ (vi) $f(x)=\dfrac{1}{x^2-x}$ (vii) $f(x)=\dfrac{x+3}{2x+1}$

Solution *Try yourself.*

Example-5 Find the natural domain of the functions

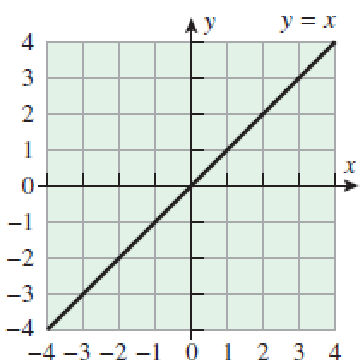
(i) $f(x)=\ln(x^2-5x+6)$ (ii) $f(x)=\sqrt{\ln\dfrac{4x-x^2}{3}}$ (iii) $f(x)=\ln\sqrt{\dfrac{5x-x^2}{4}}$ (iv) $f(x)=\dfrac{1}{\sqrt{|x|-x}}$

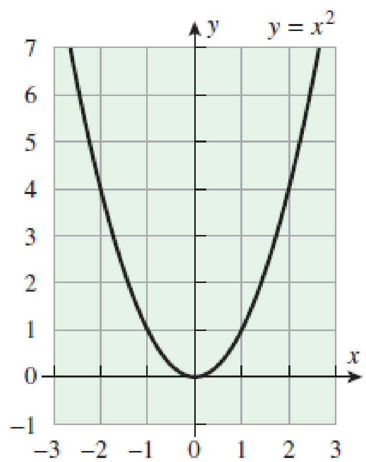
(v) $f(x)=\ln\dfrac{1-x}{1+x}$

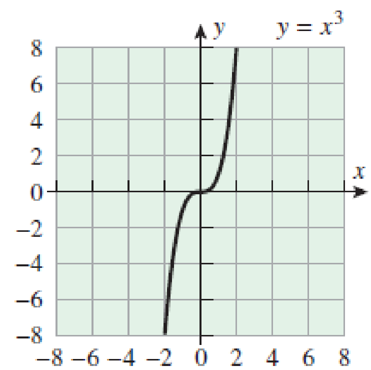
Solution *Try yourself.*

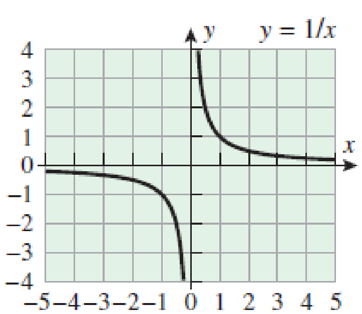
Graphs of Functions:

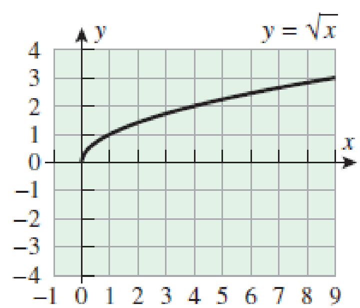
If f is a real-valued function of a real variable, then the ***graph*** of f in the xy-plane is defined to be the graph of the equation $y=f(x)$. For example, the graph of the function $f(x)=x$ is the graph of the equation $y=x$, shown in following Figure. The graphs of some other basic functions are shown below:

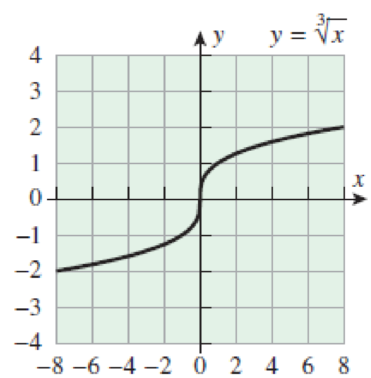












Note: Graphs can provide valuable visual information about a function. For example, since the graph of a function *f* in the *xy*-plane is the graph of the equation $y = f(x)$, the points on the graph of *f* are of the form *(x, f(x))*; that is, *the y-coordinate of a point on the graph of f is the value of f at the corresponding x-coordinate* (Figure 0.1.5). The values of *x* for which $f(x) = 0$ are the *x*-coordinates of the points where the graph of *f* intersects the *x*-axis (Figure below). These values are called the ***zeros*** of *f*, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of the graph of $y = f(x)$.

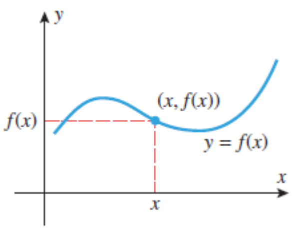


Fig 1(a): The *y*-coordinate of a point on the graph of $y = f(x)$ is the value of *f* at the corresponding *x*-coordinate.

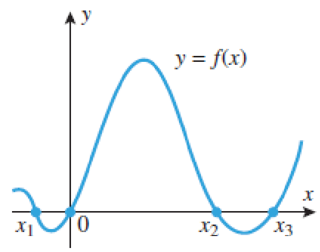


Fig 1(b): *f* has zeros at x_1, x_2 and x_3.

The vertical line test: A curve in the *xy*-plane is the graph of some function *f* if and only if no vertical line intersects the curve more than once.

Example: The graph of the equation $x^2 + y^2 = 5^2$ is a circle of radius 5 centered at the origin and hence there are vertical lines that cut the graph more than once (Figure 2(b)). Thus this equation does not define *y* as a function of *x*.

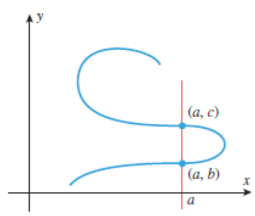


Fig-2(a): This curve cannot be the graph of a function.

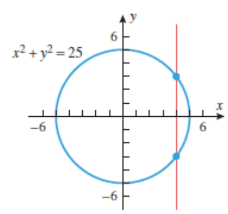


Fig-2(b): This curve cannot be the graph of a function.

Piecewise-Defined Functions:

The absolute value function $f(x) = |x|$ is an example of a function that is defined ***piecewise*** in the sense that the formula for *f* changes, depending on the value of *x*. The functions in the following examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

Example-6 A function is defined by

$$f(x) = \begin{cases} 1 - x & , \text{if } x \le -1 \\ x^2 & , \text{if } x > 1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Solution

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leqslant -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2.

$$\text{Since } -2 \leqslant -1, \text{ we have } f(-2) = 1 - (-2) = 3.$$

$$\text{Since } -1 \leqslant -1, \text{ we have } f(-1) = 1 - (-1) = 2.$$

$$\text{Since } 0 > -1, \text{ we have } f(0) = 0^2 = 0.$$

How do we draw the graph of f? We observe that if $x \leqslant -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph.

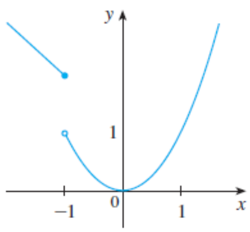


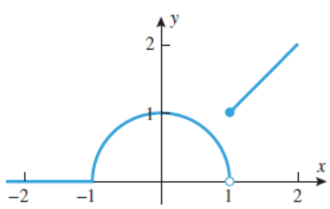
FIGURE 15

Example-7 Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0 & \text{if} \quad x \le -1 \\ \sqrt{1 - x^2} & \text{if} \quad -1 < x < 1 \\ x & \text{if} \quad x \ge 1 \end{cases}$$

▲ Figure 0.1.9

Solution. The formula for f changes at the points $x = -1$ and $x = 1$. (We call these the ***breakpoints*** for the formula.) A good procedure for graphing functions defined piecewise is to graph the function separately over the open intervals determined by the breakpoints, and then graph f at the breakpoints themselves. For the function f in this example the graph is the horizontal ray $y = 0$ on the interval $(-\infty, -1]$, it is the semicircle $y = \sqrt{1 - x^2}$ on the interval $(-1, 1)$, and it is the ray $y = x$ on the interval $[1, +\infty)$. The formula for f specifies that the equation $y = 0$ applies at the breakpoint -1 [so $y = f(-1) = 0$], and it specifies that the equation $y = x$ applies at the breakpoint 1 [so $y = f(1) = 1$]. The graph of f is shown in Figure 0.1.10. ◀

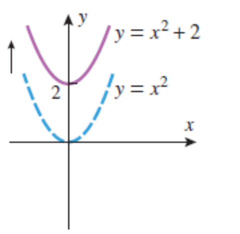
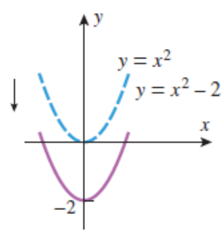
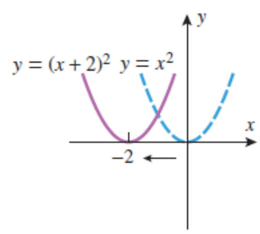
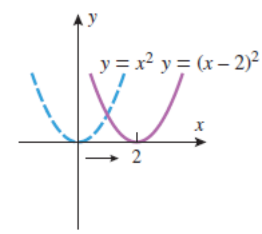


▲ Figure 0.1.10

TRANSLATIONS

The following Table illustrates the geometric effect on the graph of $y = f(x)$ of adding or subtracting a *positive* constant c to f or to its independent variable x. For example, the first result in the table illustrates that adding a positive constant c to a function f adds c to each y-coordinate of its graph, thereby shifting the graph of f up by c units. Similarly, subtracting c from f shifts the graph down by c units. On the other hand, if a positive constant c is added to x, then the value of $y = f(x + c)$ at $x - c$ is $f(x)$; and since the point $x - c$ is c units to the left of x on the x-axis, the graph of $y = f(x + c)$ must be the graph of $y = f(x)$ shifted left by c units. Similarly, subtracting c from x shifts the graph of $y = f(x)$ right by c units.

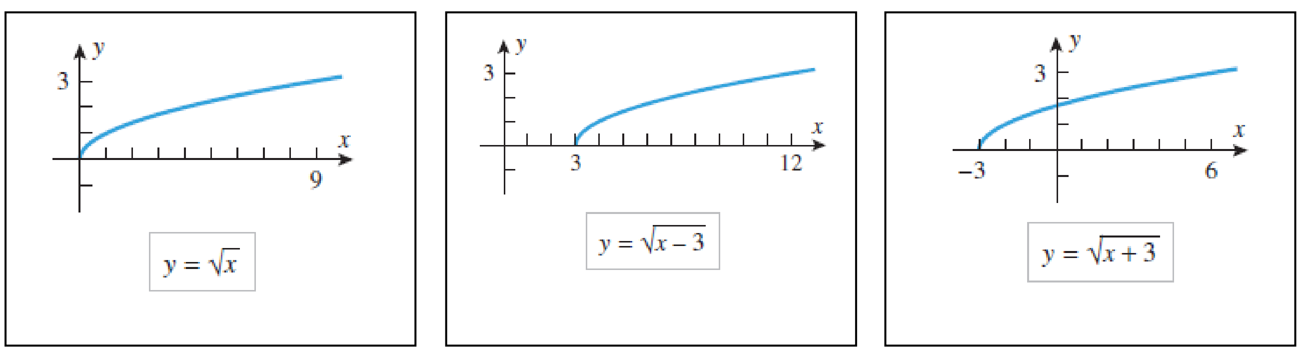
TRANSLATION PRINCIPLES

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				

Example-8 Sketch the graph of the functions:

(i) $y = \sqrt{x-3}$ (ii) $y = \sqrt{x+3}$.

Solution Using the translation principles, the graph of the equation $y = \sqrt{x-3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units. The graph of $y = \sqrt{x+3}$ can be obtained by translating the graph of $y = \sqrt{x}$ left 3 units (see the following figures).

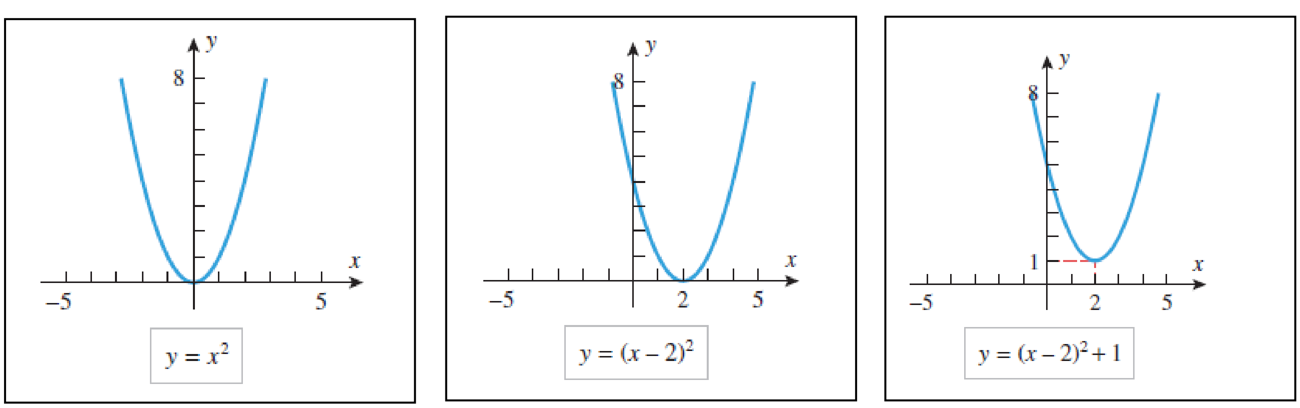


Example-9 Sketch the graph of the functions: (i) $y = x^2 - 4x + 5$. (ii) $y = x^2 + 6x + 10$.

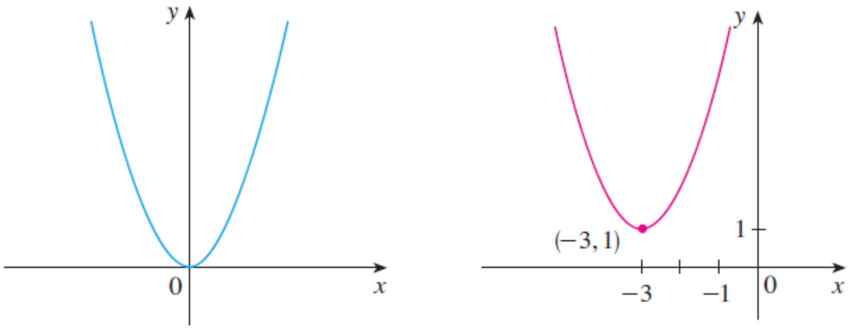
Solution (i) Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) + 1 = (x-2)^2 + 1$$

In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$.



(ii)

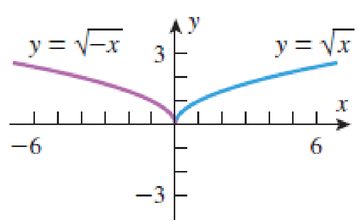
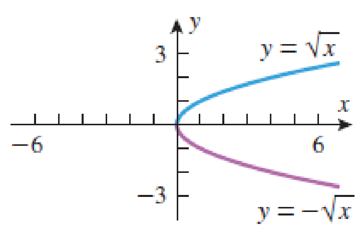


(a) $y = x^2$ (b) $y = (x+3)^2 + 1$

REFLECTIONS

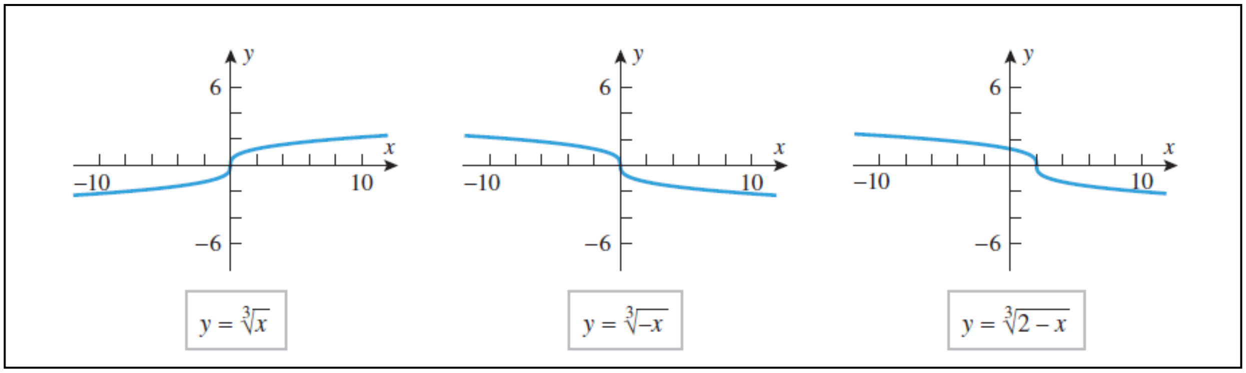
The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(-x, y)$. Similarly, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(x, -y)$ [the equation $y = -f(x)$ is equivalent to $-y = f(x)$]. This is summarized in the following Table:

REFLECTION PRINCIPLES

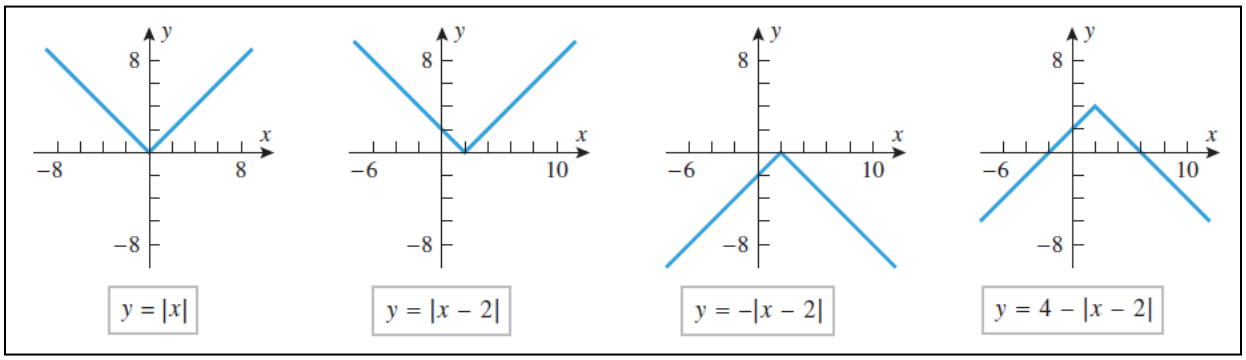
OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

Example-10 Sketch the graph of the functions: (i) $y = \sqrt[3]{2-x}$. (ii) $y = 4 - |x-2|$.

Solution (i) Using the translation and reflection principles , we can obtain the graph by a reflection followed by a translation as follows: First reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x-2)} = \sqrt[3]{2-x}$ (See the Figure).



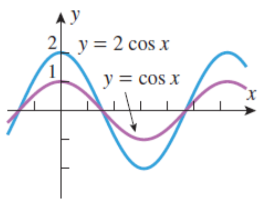
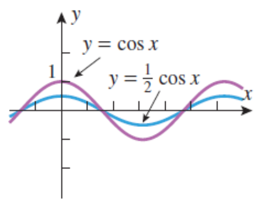
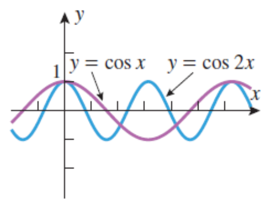
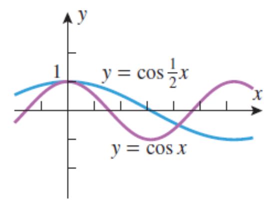
(ii) The graph can be obtained by a reflection and two translations: First translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x-2|$; then reflect this graph about the x-axis to obtain the graph of $y = -|x-2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x-2| + 4 = 4 - |x-2|$ (See the Figure).



STRETCHES AND COMPRESSIONS

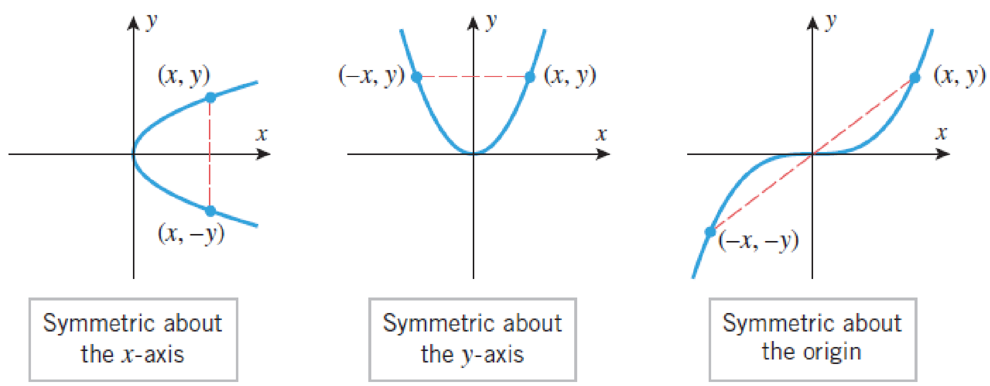
Multiplying $f(x)$ by a *positive* constant c has the geometric effect of stretching the graph of $y = f(x)$ in the y-direction by a factor of c if $c > 1$ and compressing it in the y direction by a factor of $1/c$ if $0 < c < 1$. For example, multiplying $f(x)$ by 2 doubles each y-coordinate, thereby stretching the graph vertically by a factor of 2, and multiplying by $1/2$ cuts each y-coordinate in half, thereby compressing the graph vertically by a factor of 2. Similarly, multiplying x by a *positive* constant c has the geometric effect of compressing the graph of $y = f(x)$ by a factor of c in the x-direction if $c > 1$ and stretching it by a factor of $1/c$ if $0 < c < 1$. [If this seems backwards to you, then think of it this way: The value of $2x$ changes twice as fast as x, so a point moving along the x-axis from the origin will only have to move half as far for $y = f(2x)$ to have the same value as $y = f(x)$, thereby creating a horizontal compression of the graph.] All of this is summarized in the following Table:

STRETCHING AND COMPRESSING PRINCIPLES

OPERATION ON $y = f(x)$	Multiply $f(x)$ by c $(c > 1)$	Multiply $f(x)$ by c $(0 < c < 1)$	Multiply x by c $(c > 1)$	Multiply x by c $(0 < c < 1)$
NEW EQUATION	$y = cf(x)$	$y = cf(x)$	$y = f(cx)$	$y = f(cx)$
GEOMETRIC EFFECT	Stretches the graph of $y = f(x)$ vertically by a factor of c	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$
EXAMPLE				

SYMMETRY

The following Figure illustrates three types of symmetries: ***symmetry about the x-axis***, ***symmetry about the y-axis***, and ***symmetry about the origin***. As illustrated in the figure, a curve is symmetric about the x-axis if for each point (x, y) on the graph the point $(x,-y)$ is also on the graph, and it is symmetric about the y-axis if for each point (x, y) on the graph the point $(-x, y)$ is also on the graph. A curve is symmetric about the origin if for each point (x, y) on the graph, the point $(-x,-y)$ is also on the graph. (Equivalently, a graph is symmetric about the origin if rotating the graph 180◦ about the origin leaves it unchanged.) This suggests the following symmetry tests.



Theorem ***(Symmetry Tests)***

(a) A plane curve is symmetric about the y-axis if and only if replacing x by −x in its equation produces an equivalent equation.
(b) A plane curve is symmetric about the x-axis if and only if replacing y by −y in its equation produces an equivalent equation.
(c) A plane curve is symmetric about the origin if and only if replacing both x by –x and y by −y in its equation produces an equivalent equation.

Polynomials:

A function $P(x)$ is of the form

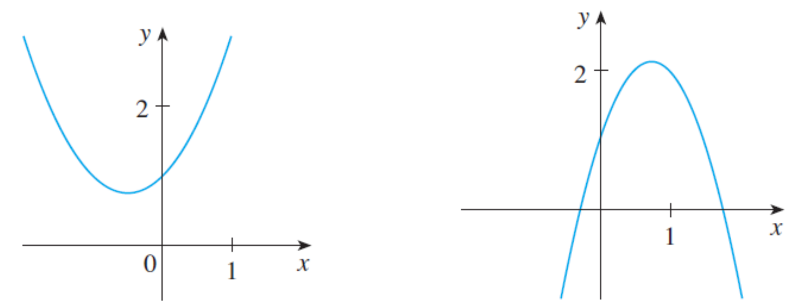
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial,** where n is a nonnegative integer and the numbers a_n, a_{n-1}, a_{n-2}, …, a_2, a_1, a_0 are constants called the **coefficients** of the polynomial. The domain of any polynomial is $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 3x^5 + x^4 - \sqrt{2}x^2 + \tfrac{2}{3}x + 9$$

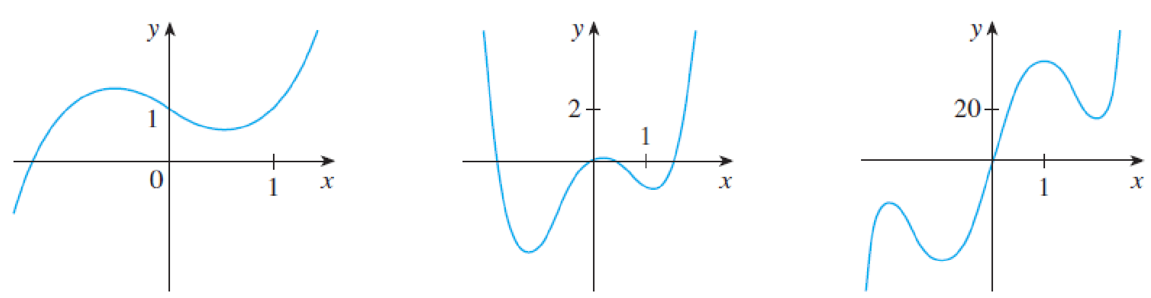
is a polynomial of degree 5.

A polynomial of degree 1 is of the form $P(x) = ax + b$, $a \neq 0$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$, $a \neq 0$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$ (See the following Figure).



(a) $y = x^2 + x + 1$ (b) $y = -2x^2 + 3x + 1$

A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$ and is called a **cubic function**. The following Figure shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). Polynomials of degree 4 and 5 are described as **quartic**, and **quintic function** respectively.



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Rational Functions:

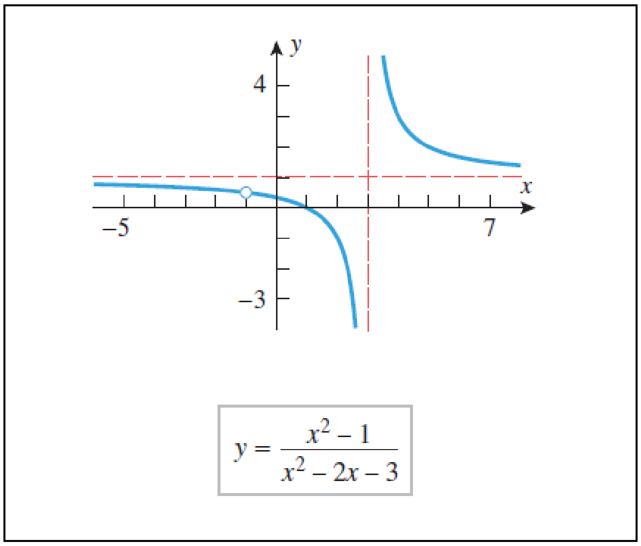
A function that can be expressed as a ratio of two polynomials is called a ***rational function***. If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

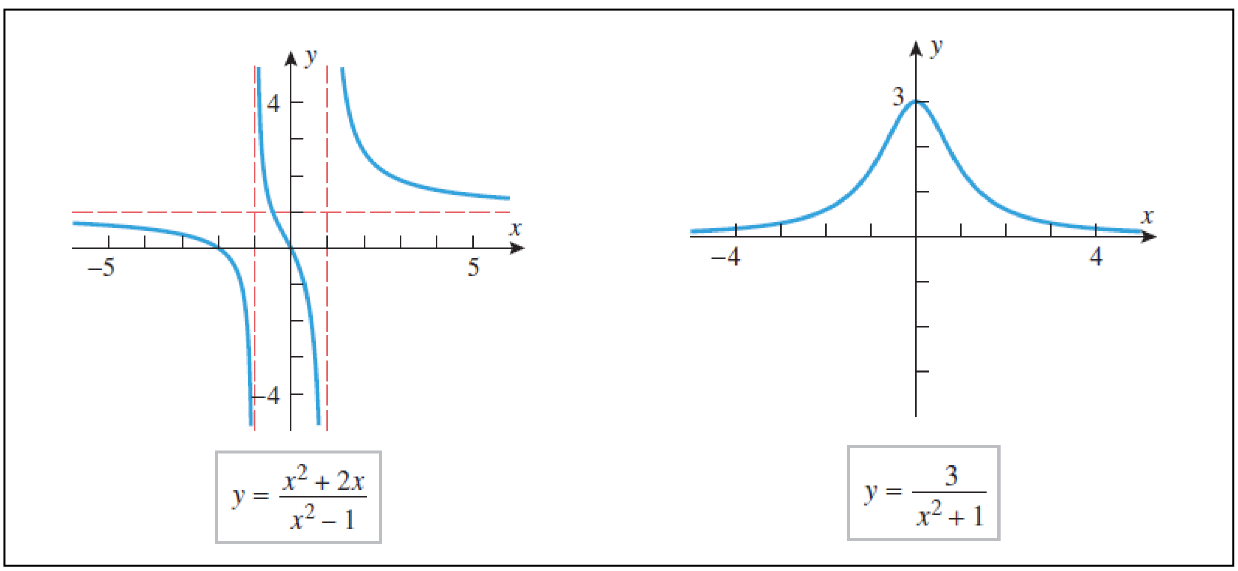
consists of all values of x such that $Q(x) \neq 0$. For example, the domain of the rational function

$$y = \frac{x^2 - 1}{x^2 - 2x - 3}$$

consists of all values of x, except $x = -1$ and $x = 3$. The following is the graph of the above rational function:



The graph of the two rational functions are as follows:



Notes:

i) The graphs of rational functions have discontinuities at the points where the denominator is zero.

ii) Rational functions may have numbers at which they are not defined. Near such points, many rational functions have graphs that closely approximate a vertical line, called a ***vertical asymptote***.

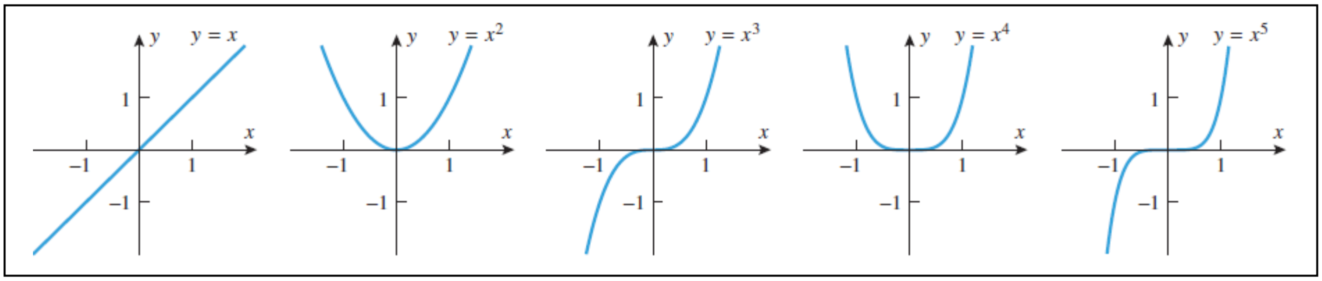
iii) The graphs of many rational functions eventually get closer and closer to some horizontal line, called a ***horizontal asymptote***, as one traverses the curve in either the positive or negative direction.

Power Functions: The Family $y = x^n$:

A function of the form $f(x) = x^a$, where a is constant, is called a ***power function***. We consider several cases.

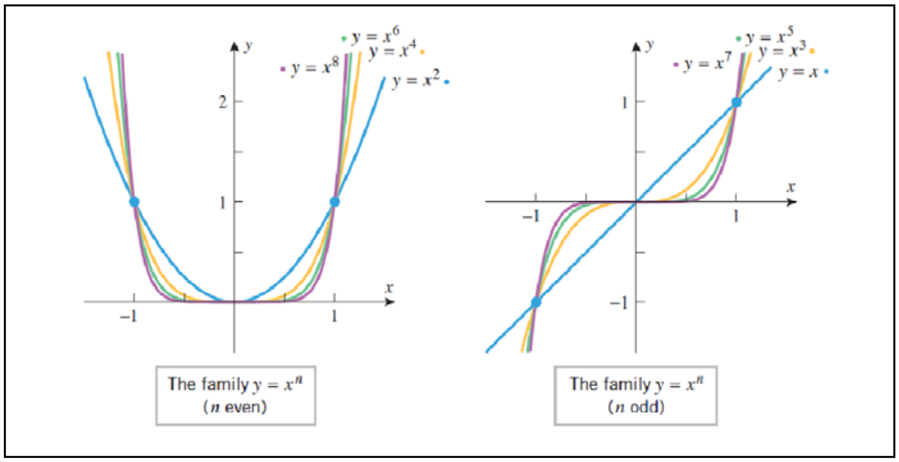
Case (i): $a = n$ **, where *n* is a positive integer:**

The graphs of the curves $y = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in the following Figure. The first graph is the line through the origin with slope, and the second is a parabola that opens up and has its vertex at the origin.



For $n \geq 2$ the shape of the curve $y = x^n$ depends on whether n is even or odd:

- For even values of n, the functions $y = x^n$ are even, so their graphs are symmetric about the *y*-axis. The graphs is similar to the parabola $y = x^2$, and each graph passes through the points (−1, 1), (0, 0), and (1, 1). As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.
- For odd values of *n*, the functions $y = x^n$ are odd, so their graphs are symmetric about the origin. The graphs is similar to that of the curve $y = x^3$, and each graph passes through the points (−1, −1), (0, 0), and (1, 1). As *n* increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.

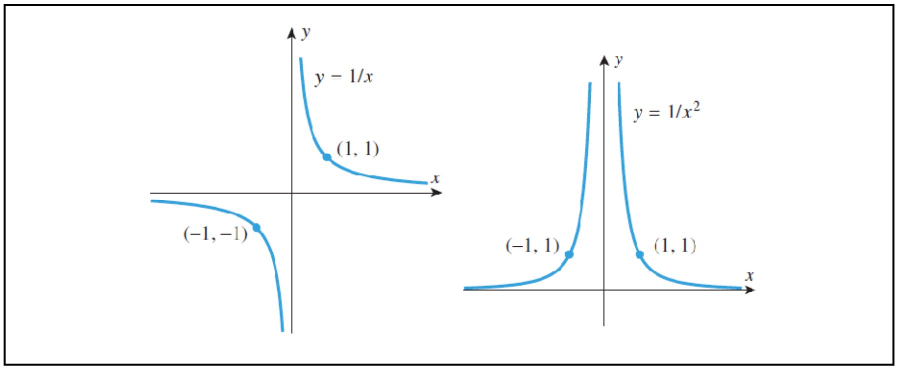


The family $y = x^n$ (n even)

The family $y = x^n$ (n odd)

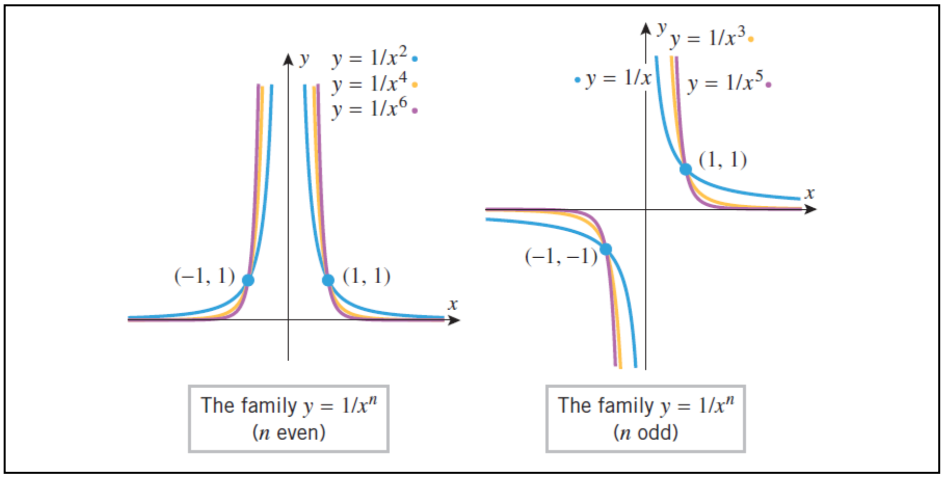
Case (ii): $a = -n$**, where *n* is a positive integer:**

If a is a negative integer, say $a = -n$, then the power functions $f(x) = x^a$ have the form $f(x) = x^{-n} = 1/x^n$. The following figures shows the graphs of $y = 1/x$ and $y = 1/x^2$. The graph of $y = 1/x$ is called an ***equilateral hyperbola***.



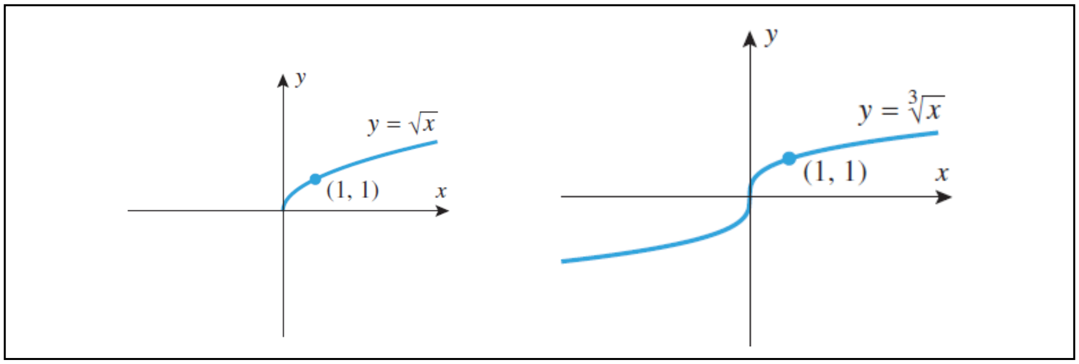
For $n \geq 2$ the shape of the curve $y = 1/x^n$ depends on whether n is even or odd:

- For even values of n, the functions $y = 1/x^n$ are even, so their graphs are symmetric about the *y*-axis. The graphs is similar to the parabola $y = 1/x^2$, and each graph passes through the points (−1, 1), and (1, 1). As n increases, the graphs become steeper over the interval $-1 < x < 0$ and $0 < x < 1$ and flatter over the intervals $x > 1$ and $x < -1$.
- For odd values of *n*, the functions $y = 1/x^n$ are odd, so their graphs are symmetric about the origin. The graphs is similar to that of the curve $y = 1/x$, and each graph passes through the points (1, 1), and (−1, −1). As *n* increases, the graphs become steeper over the interval $-1 < x < 0$ and $0 < x < 1$ and flatter over the intervals $x > 1$ and $x < -1$.
- For both even and odd values of *n* the graph $y = 1/x^n$ has a break at the origin, called a ***discontinuity***, which occurs because division by zero is undefined.



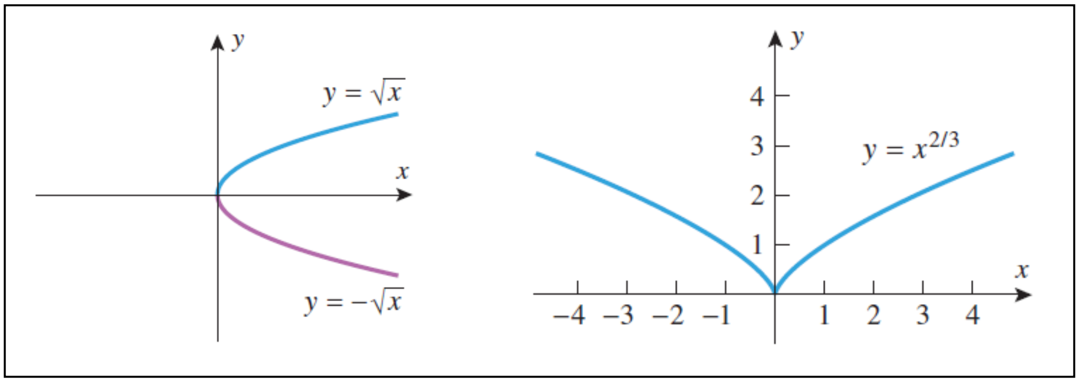
Case (iii): $a = 1/n$ **, where *n* is a positive integer (power functions with non-integer exponents):**

If $a = 1/n$, where n is a positive integer, then the power functions $f(x) = x^a$ have the form $f(x) = x^{1/n} = \sqrt[n]{x}$. The following figures shows the graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$.



Since every real number has a real cube root, the domain of the function $f(x) = \sqrt[3]{x}$ is $(-\infty, \infty)$, and hence the graph of $y = \sqrt[3]{x}$ extends over the entire *x*-axis. In contrast, the graph of $y = \sqrt{x}$ extends only over the interval $[0, \infty)$ because $\sqrt{x}$ is imaginary for negative *x*. The graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ form the upper and lower halves of the parabola $x = y^2$ (figure below). In general, the graph of $y = \sqrt[n]{x}$ extends over the entire *x*-axis if *n* is odd, but extends only over the interval $[0, \infty)$ if *n* is even.

Power functions can have other fractional exponents. For example $y = x^{2/3}$, whose graph is as follows:



Inverse Proportions:

A variable *y* is said to be ***inversely proportional to a variable x*** if there is a positive constant *k*, called the ***constant of proportionality***, such that

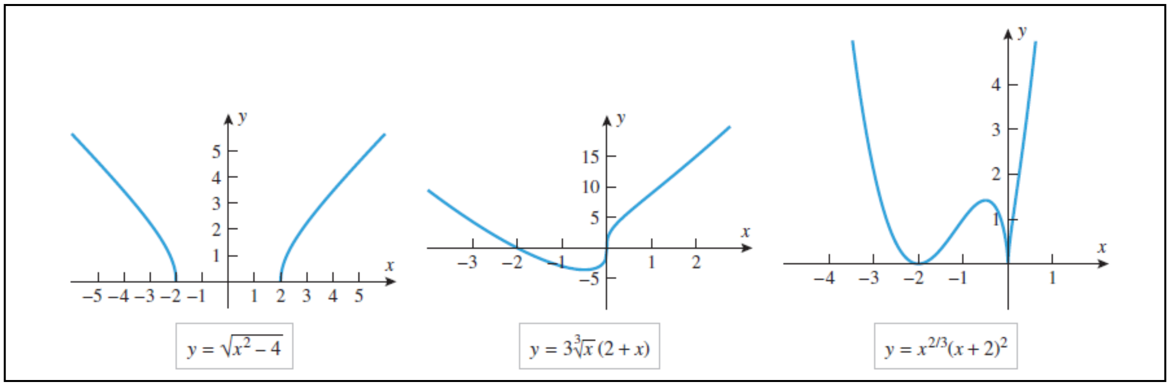
$$y = \frac{k}{x} \qquad \ldots(1)$$

Since k is assumed to be positive, the graph of (1) has the same shape as $y = 1/x$ but is compressed or stretched in the *y*-direction. Also, it should be evident from (1) that doubling x multiplies y by $1/2$, tripling x multiplies y by $1/3$, and so forth. Equation (1) can be expressed as $xy = k$, which tells us that the product of inversely proportional variables is a positive constant. This is a useful form for identifying inverse proportionality in experimental data.

Algebraic Functions:

A function is called an **algebraic function** if it can be constructed from polynomials using finitely many algebraic operations (addition, subtraction, multiplication, division, and root extraction). Any rational function is automatically an algebraic function. The following are some examples of algebraic functions and its graphs:

$f(x) = \sqrt{x^2 - 4}$, $\quad f(x) = 3\sqrt[3]{x}(2 + x)$, $\quad f(x) = x^{2/3}(x + 2)^2$



The graphs of algebraic functions vary widely, so it is difficult to make general statements about them.

Exponential and Logarithmic Functions:

The Family of Exponential Functions:

A function of the form $f(x) = b^x$, where $b > 0$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \qquad f(x) = \left(\frac{1}{2}\right)^x, \qquad f(x) = \pi^x$$

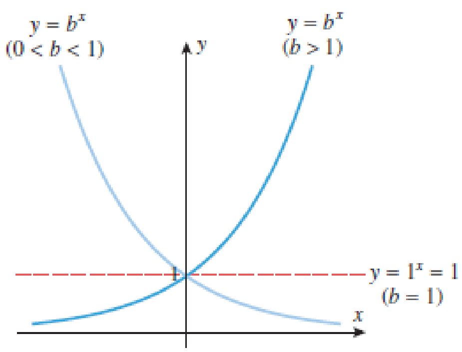
Note that an exponential function has a constant base and variable exponent. The following Figure illustrates that the graph of $y = b^x$ has one of three general forms, depending on the value of b. The graph of $y = b^x$ has the following properties:

- The graph passes through (0, 1) because $b^0 = 1$.
- If $b > 1$, the value of b^x increases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x increase indefinitely. If you traverse the graph from right to left, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x.
- If $0 < b < 1$, the value of b^x decreases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x. If you traverse the graph from right to left, the values of b^x increase indefinitely.
- If $b = 1$, then the value of b^x is constant.

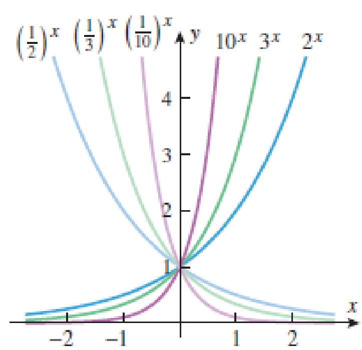
Some typical members of the family of exponential functions are graphed in Figure 0.5.2. This figure illustrates that the graph of $y = (1/b)^x$ is the reflection of the graph of $y = b^x$ about the y-axis. This is because replacing x by $-x$ in the equation $y = b^x$ yields

$$y = b^{-x} = (1/b)^x$$

The figure also conveys that for $b > 1$, the larger the base b, the more rapidly the function $f(x) = b^x$ increases for $x > 0$.



▲ **Figure 0.5.1**



▲ **Figure 0.5.2** The family $y = b^x\ (b > 0)$

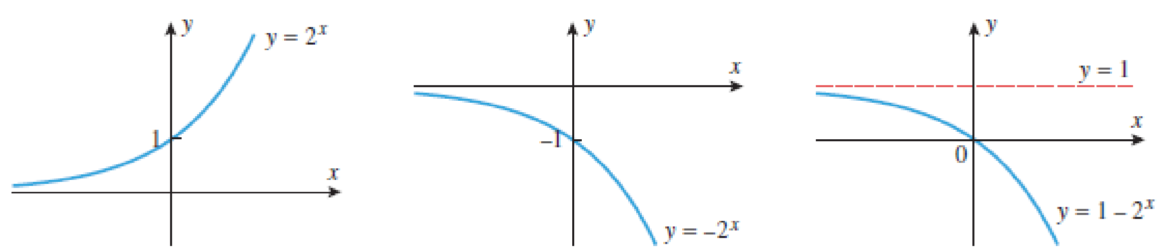
The domain and range of the exponential function $f(x) = b^x$ can also be found by examining Figure 0.5.1:

- If $b > 0$, then $f(x) = b^x$ is defined and has a real value for every real value of x, so the natural domain of every exponential function is $(-\infty, +\infty)$.
- If $b > 0$ and $b \neq 1$, then as noted earlier the graph of $y = b^x$ increases indefinitely as it is traversed in one direction and decreases toward zero but never reaches zero as it is traversed in the other direction. This implies that the range of $f(x) = b^x$ is $(0, +\infty)$.*

Example-11 Sketch the graph of the function $f(x) = 1 - 2^x$ and find its domain and range.

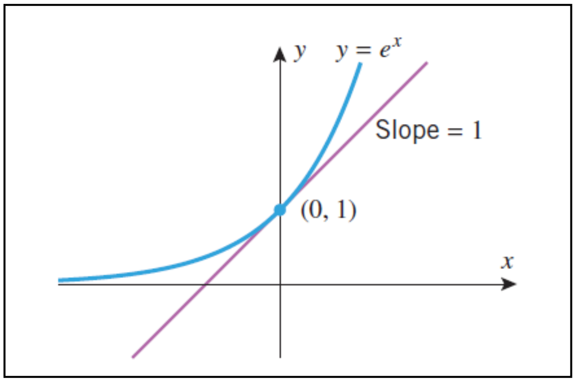
Solution

Solution. Start with a graph of $y = 2^x$. Reflect this graph across the x-axis to obtain the graph of $y = -2^x$, then translate that graph upward by 1 unit to obtain the graph of $y = 1 - 2^x$ (Figure 0.5.3). The dashed line in the third part of Figure 0.5.3 is a horizontal asymptote for the graph. You should be able to see from the graph that the domain of f is $(-\infty, +\infty)$ and the range is $(-\infty, 1)$. ◀



The Natural Exponential Function:

The function $f(x) = e^x$ is called the ***natural exponential function***. Where e, is a certain irrational number whose value to six decimal places is $e \approx 2.718282$. The graph of the function is follows:



Logarithmic Functions:

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x$$

(read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x. Thus, for example,

$$\log_{10} 100 = 2, \quad \log_{10}(1/1000) = -3, \quad \log_2 16 = 4, \quad \log_b 1 = 0, \quad \log_b b = 1$$

$$10^2 = 100 \qquad 10^{-3} = 1/1000 \qquad 2^4 = 16 \qquad b^0 = 1 \qquad b^1 = b$$

We call the function $f(x) = \log_b x$ the ***logarithmic function with base b***.

Logarithmic functions can also be viewed as inverses of exponential functions. To see why this is so, observe from Figure 0.5.1 that if $b > 0$ and $b \neq 1$, then the graph of $f(x) = b^x$ passes the horizontal line test, so b^x has an inverse. We can find a formula for this inverse with x as the independent variable by solving the equation

$$x = b^y$$

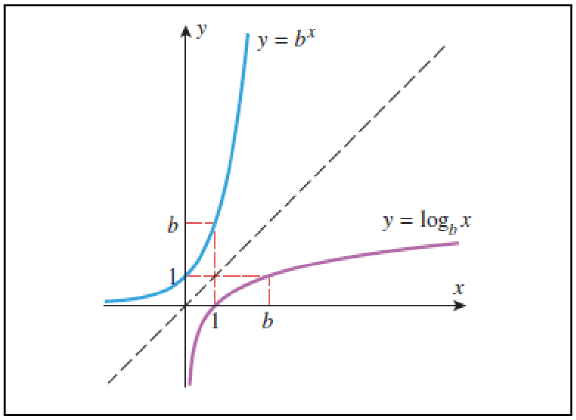
for y as a function of x. But this equation states that y is the logarithm to the base b of x, so it can be rewritten as

$$y = \log_b x$$

Thus, we have established the following result.

0.5.1 THEOREM *If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.*

It follows from this theorem that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$ (see Figure 0.5.6 for the case where $b > 1$). Figure 0.5.7 shows the graphs of $y = \log_b x$ for various values of b. Observe that they all pass through the point $(1, 0)$.



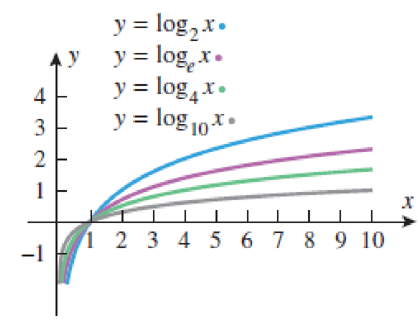


Figure 0.5.7 The family $y = \log_b x \ (b > 1)$

The most important logarithms in applications are those with base e. These are called ***natural logarithms*** because the function $\log_e x$ is the inverse of the natural exponential function e^x. It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than $\log_e x$. For example,

$$\ln 1 = 0, \qquad \ln e = 1, \qquad \ln 1/e = -1, \qquad \ln(e^2) = 2$$

Since $e^0 = 1$ | Since $e^1 = e$ | Since $e^{-1} = 1/e$ | Since $e^2 = e^2$

In general,

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$

As shown in Table 0.5.3, the inverse relationship between b^x and $\log_b x$ produces a correspondence between some basic properties of those functions.

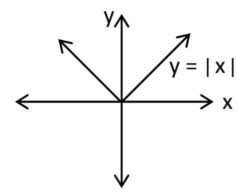
Table 0.5.3

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

Even functions: A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)$ for all $x \in D_f$.

For example, each of the following is an even function:



(i) If $f(x) = x^2$ then $f(-x) = (-x)^2 = x^2 = f(x)$.

(ii) If $f(x) = \cos x$, then $f(-x) = \cos(-x) = \cos x = f(x)$.

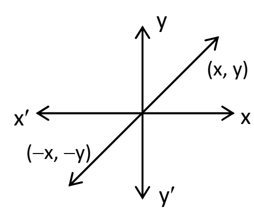
(iii) If $f(x) = |x|$, then $f(-x) = |-x| = |x| = f(x)$.

The graph of this even function (modulus function) is shown in the above figure:

✪ Note : The above graph is symmetrical about y-axis.

Odd functions: A function $y = f(x)$ is said to be an odd function if if $f(-x) = -f(x)$ for all $x \in D_f$.

For example, each of the following is an odd function:



(i) If $f(x) = x^3$ then $f(-x) = (-x)^3 = -x^3 = -f(x)$.

(ii) If $f(x) = \sin x$, then $f(-x) = \sin(-x) = -\sin x = -f(x)$.

Graph of the odd function y = x is shown in the above graph:

✪ Note: The above graph is symmetrical about origin.

Properties of odd and even functions:

1. Inverse of an even function is not defined.
2. Every function can be expressed as the sum of an even and an odd function.
   i.e.,
   $$f(x) = \frac{1}{2}\{f(x) + f(-x)\} + \frac{1}{2}\{f(x) - f(-x)\}$$
   = {Even Function} + {odd function}
3. If $f(x) - f(-x) = 0$ then $f(x)$ is an even function and if $f(x) + f(-x) = 0$ then $f(x)$ is an odd function.
4. A function may neither be even nor odd.
5. $f(x) = 0$ is the only function which is defined on the entire number line is even and odd at the same time.
6. Every odd continuous function passes through origin.
7. Every even function $y = f(x)$ is not one-one $\forall\, x \in D_f$
8. The derivative of an odd function is an even function and derivative of an even function is an odd function.
9. If f and g both are even or any one of them is odd then fog will be even. If f and g both are odd then gof is odd.
10. The square of an even or an odd function is always an even function.
11. The graph of an even function is symmetrical about the Y-axis.
12. The graph of an odd function is symmetrical about the origin.
13. Table of two functions which are attached:

$f(x)$	$g(x)$	$f(x)+g(x)$	$f(x)-g(x)$	$f(x)\,g(x)$	$\frac{f(x)}{g(x)}$	(fog) x
Even	Even	Even	Even	Even	Even	Even
Even	Odd	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Even	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Odd	Odd	Odd	Even	Even	Odd

Example

$\frac{f(x)+f(-x)}{2}[g(x)-g(-x)]$ is odd function

1. $x.\frac{e^x+1}{e^x-1}$ even.
2. $\log\left[x+\sqrt{1+x^2}\right]$ odd
3. $\log\frac{(1-x)}{1+x}$ odd function
4. $\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$ odd function.
5. $x\frac{a^x+1}{a^x-1}$ even function
6. $\frac{e^x+1}{e^x-1}$ odd function
7. $-x\,|\,x\,|$ odd function.
8. $\frac{a^x-1}{a^x+1}$ is odd function
9. $|\,x\,|$ even
   Following are examples of neither even nor odd function: $(x^2+x : \sin x + \cos x : e^x . [x]$, $|x-2|$ etc.

Example-12 Show that the functions (i) $f(x)=\log(x^3+\sqrt{1+x^6})$ (ii) $f(x)=\log\left(\frac{1+x}{1-x}\right)$ are odd functions.

Solution *Try yourself.*